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Complicated dynamics of a ring neural network with time delays

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Abstract
This paper studies the dynamical behavior of a ring neural network with time delays. On the basis of Lyapunov's method, the asymptotic stability of the equilibrium is first investigated, and the delay-dependent criteria ensuring global stability for the ring neural network are obtained. Moreover, based on the global Hopf bifurcation theorem for FDE, the conditions that guarantee the global existence of periodic solutions are determined. It shows that periodic solutions bifurcating from the trivial equilibrium can continue when the time delay is far away from the critical value. Some examples are induced to illustrate our results. In addition, complicated dynamics of the model are investigated with the help of numerical simulation. The study results show that the model exhibits period-doubling bifurcations which lead eventually to chaos; and the chaos can also directly occur via the bifurcations from the quasi-periodic solutions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Over the past decades, there has been an increasing interest in the study of neuron systems such as in the study of their mathematical modeling and artificial representations. Researchers have found neural networks that have many applications in different areas such as pattern recognition, associative memory and combinatorial optimization. Such applications heavily depend on the dynamical behavior. Thus, the analysis of the dynamical behavior is a necessary step for the practical design of neural networks. One of the most investigated problems in dynamical behavior of neural networks is the global asymptotic stability for the equilibrium. For example, when the neural networks are adopted as parallel computation and signal processing for solving optimization problems, it is required that there exists a well-defined computable solution for all possible initial states. From the mathematical and engineering viewpoint, this means that the network should have a unique and global stable equilibrium. Thus, the global stability of neural systems is of great importance for both the practical and theoretical purposes, and has been extensively investigated [1–13].

Indeed, neural networks inevitably incorporate time delays since the transmission of information between the neurons is not instantaneous. Uncontrolled delays may degrade network performance: they may interfere with information processing by making the equilibrium unstable [14–20]. Therefore, time delays usually play a destabilizing role. Recently, researchers have obtained some delay-dependent criteria for the local stability of neural network. However, a little progress has been achieved for the global stability criteria dependent of delays. Therefore, we wish to know if the time delays can be adopted as global stabilizers rather than destabilizers.

In addition, when the connection matrix of neural networks is symmetric or antisymmetric, the Hopfield network is always a convergent gradient network or a stable network in the absence of delays [21]. However, when the delays are present, the above convergence and stability properties may be lost even for very small delays, and periodic solutions may arise. Usually, these periodic solutions only exist in a small neighborhood of the critical values. Therefore, we wish to know whether these periodic solutions can continue for a large range of parameter values. It is also an important mathematical subject to investigate if these nontrivial periodic solutions exist globally.

In this paper, we take the ring neural network as the research model to illustrate the above problems. Rings networks are of a limited biological relevance, and may be regarded as building blocks for networks with more realistic connection topologies. Ring networks belong to a class of cyclic feedback systems whose dynamical behavior has been investigated in more detail. For example, Baldi and Atiya [22] proposed the following simple neural network model:

$$\dot{x}_i(t) = -x_i(t)/T_i + T_{ii-1} f(x_{i-1}(t-\tau_{ii-1})), \qquad i = 1, 2, \ldots, n, \tag{1}$$

and investigated the effects of delays on its dynamical properties.

Campbell [23] studied the neural network model

$$C_i\dot{x}_i(t) = -x_i(t)/R_i + F_i(x_i(t-\tau_s)) + G_i(x_{i-1}(t-\tau_{i-1})), \qquad i = 1, 2, \ldots, n, \tag{2}$$

and investigated the linear stability of fixed points and the existence of the co-dimension two bifurcation.

Yuan and Campbell [24, 25] studied a network with ring construction described by

$$\dot{x}_i(t) = -x_i(t) + \alpha f(x_i(t-\tau_s)) + \beta[g(x_{i-1}(t-\tau)) + g(x_{i+1}(t-\tau))]. \tag{3}$$

Huang and Wu [26] and Guo and Huang [27] studied the following ring network:

$$\dot{x}_i(t) = -x_i(t) + f(x_i(t-\tau)) - [g(x_{i-1}(t-\tau)) + g(x_{i+1}(t-\tau))]. \tag{4}$$

They analyzed the bifurcation and stability of nontrivial synchronous solutions from the trivial solution, and discussed the stability of the equivariant Hopf bifurcation. From then on, many authors have focused on the stability of fixed points, the bifurcation and existence of periodic solutions to systems (1)–(4) with a few neurons [28–35]. However, most of the papers investigate the local dynamics of ring neural networks, such as the local stability of fixed points and local Hopf bifurcation. A little progress has been achieved for the global stability criteria dependent of delays [36–38] and the global bifurcation of high dimensional neural networks, especially when the link weights matrix of neural networks is asymmetrical.

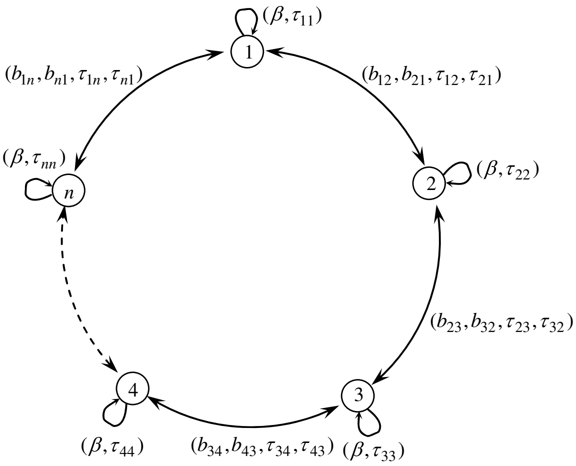

Figure 1. Architecture of a ring neural network with multiple time delays.

In this paper, we study a ring neural network with time delays modeled by

$$\dot{x}_i = -kx_i(t) + \sum_{j=i-1}^{i+1} b_{ij} f(x_j(t - \tau_{ij})), \qquad i = 1, 2, \ldots, n, \tag{5}$$

where

$$\mathbf{B} = \{b_{ij}\}_{i,j=1}^{n} = \begin{bmatrix} \beta & b_{12} & 0 & 0 & \cdots & b_{1n} \\ b_{21} & \beta & b_{23} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & & \vdots \\ 0 & 0 & & \cdots & \beta & b_{n-1,n} \\ b_{n1} & 0 & & \cdots & b_{n,n-1} & \beta \end{bmatrix}, \tag{6}$$

$k > 0$; $x_i(t)$ denotes the neuron response; $f(u) = \tanh(u)$ is the activation function of neurons; $b_{ii} = \beta \neq 0$ is the connection strength of self-feedback of neuron; $b_{ij}(i \neq j)$ denotes the connection strengths between the two neurons, $b_{i,i-1} \neq 0$, $b_{i,i+1} \neq 0$, and $b_{ij} = 0$ for $j \neq i-1, i, i+1$; index i is taken to modulo n, so that, for instance, $b_{n,n+1} = b_{n1} = b_{01}$, $x_0 = x_n$, $x_{n+1} = x_1$; the delays τ_{ij} is non-negative. The architecture of this model is illustrated in figure 1.

The initial conditions associated with (5) are assumed to be of the form

$$x_i(t) = \phi_i(t), \qquad t \in \left[-\max_{1 \leqslant i,j \leqslant n} \tau_{ij}, 0\right], \qquad i = 1, 2, \ldots, n. \tag{7}$$

The aim of this paper is to analyze global dynamical behavior of (5) including the delay-dependent criteria ensuring global stability and global existence of periodic solutions bifurcated from the trivial equilibrium. Special attention is paid to the complicated dynamics of the model. The paper is organized as follows. In the following section, we construct a suitable Lyapunov function to obtain the delay-dependent criteria for the global stability of (5). In sections 3 and 4, the local and global existences of multiple periodic solutions are discussed. And the routes to chaos are studied in section 5. Finally, several concluding remarks are drawn in section 6.

2. Global asymptotic stability

A number of papers deal with conditions ensuring the global stability for Hopfield-type neural network and their generalizations. According to these papers, one can obtain the delay-independent criteria for the global stability of neural network (5).

Theorem 1. *If neuronal gains and connection in (5) satisfy the inequality*

$$-k + |\beta| + |b_{i-1,i}| + |b_{i+1,i}| < 0, \qquad \text{for all} \quad 1 \leqslant i \leqslant n, \tag{8}$$

then the zero solution is a unique and globally asymptotic stable equilibrium.

The proof of this theorem is similar to that of theorems 1 and 2 in [3]. For clarity, we give the proof of this theorem.

Proof. From (5), it is easy to see that an arbitrary solution of (5) satisfies the following inequalities:

$$-kx_i(t) - \sigma_i \leqslant \dot{x}_i(t) \leqslant -kx_i(t) + \sigma_i, \qquad i = 1, 2, \ldots, n, \tag{9}$$

where $\sigma_i = \sum_{j=i-1}^{i+1} |b_{ij}|, \quad i = 1, 2, \ldots, n.$

It will follow from (9) that the set $\Omega \subset \Re^n$ defined by

$$\Omega = \{\mathbf{x} \mid \mathbf{x} = (x_1, x_2, \ldots, x_n);\ k|x_i| \leqslant \sigma_i,\ i = 1, 2, \ldots, n\} \tag{10}$$

is invariant with respect to the delay differential equations (5). Thus, if (5) has an equilibrium, then such an equilibrium is a fixed point of the mapping $F : \Omega \to \Re^n$ defined by

$$F = \left[\sum_{j=0}^{2} \frac{1}{k} b_{1j} f(x_j(t)), \sum_{j=1}^{3} \frac{1}{k} b_{2j} f(x_j(t)), \ldots, \sum_{j=n-1}^{n+1} \frac{1}{k} b_{nj} f(x_j(t))\right]^{\mathrm{T}}. \tag{11}$$

If (8) is satisfied, then there exists a number $c \in (0, 1)$ such that the following inequality holds:

$$\max_{1\leqslant i\leqslant n} \frac{1}{k}\{|\beta| + |b_{i-1,i}| + |b_{i+1,i}|\} \leqslant c < 1. \tag{12}$$

We note from (12) that if $\mathbf{u}$ and $\mathbf{v}$ are any two points of Ω, then

$$\begin{aligned}
\|F(\mathbf{u}) - F(\mathbf{v})\| &= \left|\sum_{i=1}^{n}\sum_{j=i-1}^{i+1} \frac{1}{k} b_{ij}[f(u_j) - f(v_j)]\right| \\
&= \left|\sum_{i=1}^{n}\sum_{j=i-1}^{i+1} \frac{1}{k} b_{ij} f'(\xi_j)(u_j - v_j)\right| \leqslant \sum_{i=1}^{n}\sum_{j=i-1}^{i+1} \frac{1}{k}|b_{ij}||u_j - v_j| \\
&\leqslant \sum_{i=1}^{n}\sum_{j=i-1}^{i+1} \frac{1}{k}|b_{ji}||u_i - v_i| = \sum_{i=1}^{n}\left\{\sum_{j=i-1}^{i+1} \frac{1}{k}|b_{ji}|\right\}|u_i - v_i| \\
&\leqslant c\sum_{i=1}^{n}|u_i - v_i| \leqslant c\|\mathbf{u} - \mathbf{v}\|.
\end{aligned} \tag{13}$$

In deriving (13) and subsequent inequalities, we have used the facts that ξ_j lies between u_j and v_j as well as $0 \leqslant f'(\theta) \leqslant 1$ for any θ.

The mapping F is continuous and $F(\Omega) \subset \Omega$; it follows from (13) and $c < 1$ that F is a contraction on Ω. By the well-known contraction mapping principle, we conclude that the origin is a unique fixed point.

In order to analyze the global stability of zero solutions, we consider a Lyapunov functional $V(t)$ defined by

$$V(t)=\sum_{i=1}^{n}\left\{|x_i(t)|+\sum_{j=i-1}^{i+1}|b_{ij}|\int_{t-\tau_{ij}}^{t}|x_j(s)|\,\mathrm{d}s\right\}. \tag{14}$$

Calculating the upper right derivative $\mathrm{D}^{+}V$ of V along the solution of (5), we have

$$\begin{aligned}
\mathrm{D}^{+}V&=\sum_{i=1}^{n}\left\{\operatorname{sgn}(x_i(t))\dot{x}_i(t)+\sum_{j=i-1}^{i+1}|b_{ij}|[|x_j(t)|-|x_j(t-\tau_{ij})|]\right\}\\
&=\sum_{i=1}^{n}\left\{\operatorname{sgn}(x_i(t))\left[-kx_i(t)+\sum_{j=i-1}^{i+1}b_{ij}f(x_j(t-\tau_{ij}))\right]\right.\\
&\quad\left.+\sum_{j=i-1}^{i+1}|b_{ij}|[|x_j(t)|-|x_j(t-\tau_{ij})|]\right\}\\
&\leqslant\sum_{i=1}^{n}\left\{-k|x_i(t)|+\sum_{j=i-1}^{i+1}|b_{ij}||f(x_j(t-\tau_{ij}))|+\sum_{j=i-1}^{i+1}|b_{ij}||x_j(t)|\right.\\
&\quad\left.-\sum_{j=i-1}^{i+1}|b_{ij}||x_j(t-\tau_{ij})|\right\}.
\end{aligned} \tag{15}$$

By the differential mean-value theorem, we have

$$f(x_j(t-\tau_{ij}))=x_j(t-\tau_{ij})f'(\xi),$$

where ξ is the value between 0 and $x_j(t-\tau_{ij})$. Note that $0\leqslant f'(\xi)\leqslant 1$, we can obtain

$$|x_i(t-\tau_{ij})|=\frac{|f(x_j(t-\tau_{ij}))|}{f'(\xi)}\geqslant|f(x_j(t-\tau_{ij}))|.$$

Thus,

$$\begin{aligned}
\mathrm{D}^{+}V&\leqslant\sum_{i=1}^{n}\left\{-k|x_i(t)|+\sum_{j=i-1}^{i+1}|b_{ij}||x_j(t-\tau_{ij})|+\sum_{j=i-1}^{i+1}|b_{ij}||x_j(t)|-\sum_{j=i-1}^{i+1}|b_{ij}||x_j(t-\tau_{ij})|\right\}\\
&=\sum_{i=1}^{n}\left\{-k|x_i(t)|+\sum_{j=i-1}^{i+1}|b_{ij}||x_j(t)|\right\}.
\end{aligned}$$

From (6), it is easy to see that $b_{ij}=0$ for $j\neq i-1,i,i+1$. Then, we can rewrite the above inequality as

$$\begin{aligned}
\mathrm{D}^{+}V&\leqslant\sum_{i=1}^{n}(-k|x_i(t)|)+\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}||x_j(t)|=\sum_{i=1}^{n}(-k|x_i(t)|)+\sum_{j=1}^{n}\sum_{i=1}^{n}|b_{ji}||x_i(t)|\\
&=\sum_{i=1}^{n}(-k|x_i(t)|)+\sum_{i=1}^{n}\sum_{j=i-1}^{i+1}|b_{ji}||x_i(t)|=\sum_{i=1}^{n}\left[-k|x_i(t)|+\sum_{j=i-1}^{i+1}|b_{ji}|\right]|x_i(t)|\\
&=\sum_{i=1}^{n}(-k+|b_{i-1,i}|+|\beta|+|b_{i+1,i}|)|x_i(t)|.
\end{aligned}$$

Now, by the standard Lyapunov theorem in functional differential equations, if (8) is satisfied, then $D^+V < 0$ and the trivial solution of (5) is global asymptotically stable. This completes the proof.

Theorem 1 investigates the globally stable criteria independent of time delays. In this case, delays only affect the convergence rate. However, there are cases that delays play the key roles in discussion of whether or not a system is stable. Hence, it is needed to analyze how time delays affect the global stability. In [36, 37], the authors studied the neurons networks with $n = 2$ and obtained some delay-dependent criteria for the global stability of equilibrium. In this paper, we will extend their results to the case with any n neurons.

First, by (5), we know that

$$|\dot{x}_i(t) + kx_i(t)| \leqslant \sum_{j=i-1}^{i+1} |b_{ij}||f(x_j(t-\tau_{ij}))| \leqslant |\beta| + |b_{i,i-1}| + |b_{i,i+1}|, \qquad i = 1, 2, \ldots, . \tag{16}$$

Then, it is clear from (5) and (16) that

$$\lim_{t\to\infty} \sup |x_i(t)| \leqslant (|\beta| + |b_{i,i-1}| + |b_{i,i+1}|)/k \equiv L_i. \tag{17}$$

Thus, for sufficiently large $T > 0$, we approximately have $|x_i(t)| \leqslant L_i$ when $t \geqslant T$.

Theorem 2. *If the neuronal gains and connection weights in (5) satisfy the inequality*

$$\max_{1\leqslant i\leqslant n} \left\{ -k + \beta + \frac{1}{2} \sum_{j=i-1, j\neq i}^{i+1} (|b_{ij}| + |b_{ji}|) + \alpha_i \right\} < 0, \tag{18}$$

then the zero solutions of (5) is globally asymptotically stable, where

$$\begin{cases} \alpha_i = \dfrac{1}{2}\left[\displaystyle\sum_{j=i-1}^{i+1} (|b_{ij}|\tau_{ij}q_{ij}) + \sum_{j=i-1}^{i+1} \left(\left|b_{ji}\tau_{ji}k/p_i^2\right|\right) + \sum_{s=i-1}^{i+1}\sum_{j=s-1}^{s+1} (|b_{sj}|b_{ji}|\tau_{sj}|)\right] \\ p_i = f'(L_i) \\ q_{ij} = k + \displaystyle\sum_{s=i-1}^{i+1} |b_{js}|. \end{cases}$$

Proof. We first consider a Lyapunov function $W = W(x_1, x_2, \ldots, x_n)$ for (5) defined by

$$W(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \int_0^{x_i(t)} f(\xi)\, \mathrm{d}\xi. \tag{19}$$

It is easy to see that W is continuous and non-negative for any $x_1, x_2, \ldots, x_n \in \Re$, and the upper right derivative D^+W of W along the solution of (5) satisfies

$$\begin{aligned} D^+W\big|_{(5)} &= \sum_{i=1}^{n} f(x_i(t))\dot{x}_i(t) \\ &= \sum_{i=1}^{n} f(x_i(t)) \left[-kx_i(t) + \sum_{j=i-1}^{i+1} b_{ij} f(x_j(t-\tau_{ij})) \right] \\ &= \sum_{i=1}^{n} f(x_i(t)) \left\{ -kx_i(t) + \sum_{j=i-1}^{i+1} b_{ij}[f(x_j(t-\tau_{ij})) - f(x_j(t)) + f(x_j(t))] \right\} \end{aligned}$$

$$= \sum_{i=1}^{n} f(x_i(t)) \left\{ -kx_i(t) + \sum_{j=1-1}^{i+1} b_{ij}[f(x_j(t)) + A_j] \right\}$$
$$= \sum_{i=1}^{n} -kx_i(t) f(x_i(t)) + \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} b_{ij} f(x_j(t)) f(x_i(t)) + \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} b_{ij} f(x_i(t)) A_j, \tag{20}$$

where $A_j = \int_t^{t-\tau_{ij}} f'(x_j(\xi))\dot{x}_j(\xi)\,\mathrm{d}\xi$.

The second item of the right side of the above equation can be written as

$$\sum_{i=1}^{n} \sum_{j=i-1}^{i+1} b_{ij} f(x_j(t)) f(x_i(t)) = \sum_{i=1}^{n} \left\{ b_{ii} f^2(x_i(t)) + \sum_{j=i-1, j\neq i}^{i+1} b_{ij} f(x_j(t)) f(x_i(t)) \right\}$$
$$\leqslant \sum_{i=1}^{n} \left\{ b_{ii} f^2(x_i(t)) + \frac{1}{2} \sum_{j=i-1, j\neq i}^{i+1} |b_{ij}|[f^2(x_i(t)) + f^2(x_j(t))] \right\}$$
$$= \sum_{i=1}^{n} \left[b_{ii} + \frac{1}{2} \sum_{j=i-1, j\neq i}^{i+1} (|b_{ij}| + |b_{ji}|) \right] f^2(x_i(t)).$$

Substituting the facts $x_i(t) f(x_i(t)) \geqslant f^2(x_i(t))$ into the above two equations give

$$\mathrm{D}^+ W\big|_{(5)} \leqslant \left[-k + b_{ii} + \frac{1}{2} \sum_{j=i-1, j\neq i}^{i+1} (|b_{ij}| + |b_{ji}|) \right] f^2(x_i(t)) + \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} |b_{ij}||f(x_i(t)A_j)|$$
$$= \Phi_i f^2(x_i(t)) + \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} |b_{ij}||f(x_i(t)A_j)|,$$

where $\Phi_i = -k + b_{ii} + \frac{1}{2} \sum_{j=i-1, j\neq i}^{i+1} (|b_{ij}| + |b_{ji}|)$.

Using (5), we have

$$|f(x_i(t))A_j| = \left| f(x_i(t)) \int_{t-\tau_{ij}}^{t} f'(x_j(\xi))\dot{x}_j(\xi)\,\mathrm{d}\xi \right| \leqslant |f(x_i(t))| \int_{t-\tau_{ij}}^{t} |\dot{x}_j(\xi)|\,\mathrm{d}\xi$$
$$\leqslant |f(x_i(t))| \int_{t-\tau_{ij}}^{t} \left\{ k|x_j(\xi)| + \sum_{s=i-1}^{i+1} |b_{js}||f(x_s(\xi - \tau_{js}))| \right\} \mathrm{d}\xi$$
$$\leqslant \frac{k}{2} \int_{t-\tau_{ij}}^{t} \left[f^2(x_i(t)) + x_j^2(\xi) \right] \mathrm{d}\xi$$
$$+ \frac{1}{2} \int_{t-\tau_{ij}}^{t} \sum_{s=i-1}^{i+1} |b_{js}|[f^2(x_i(t)) + f^2(x_s(\xi - \tau_{js}))]\,\mathrm{d}\xi$$
$$= \frac{\tau_{ij}}{2} \left[k + \sum_{s=i-1}^{i+1} |b_{js}| \right] f^2(x_i(t))$$
$$+ \frac{1}{2} \int_{t-\tau_{ij}}^{t} \left\{ kx_j^2(\xi) + \sum_{s=i-1}^{i+1} |b_{js}| f^2(x_s(\xi - \tau_{js})) \right\} \mathrm{d}\xi$$
$$= \frac{\tau_{ij}}{2} q_{ij} f^2(x_i(t)) + \frac{1}{2} \int_{t-\tau_{ij}}^{t} z_{ij}(\xi)\,\mathrm{d}\xi, \tag{21}$$

where $q_{ij} = k + \sum_{s=i-1}^{i+1} |b_{js}|$ and $z_{ij}(\xi) = kx_j^2(\xi) + \sum_{s=i-1}^{i+1} |b_{js}| f^2(x_s(\xi - \tau_{js}))$.

Therefore, we have

$$\mathrm{D}^+ W|_{(5)} \leqslant \sum_{i=1}^{n} \Phi_i f^2(x_i(t)) + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} \left\{ |b_{ij}| \left[\tau_{ij} q_{ij} f^2(x_i(t)) + \int_{t-\tau_{ij}}^{t} z_{ij}(\xi)\,\mathrm{d}\xi \right] \right\}. \tag{22}$$

Now, we can define a Lyapunov functional $V = V(x_1, x_2, \ldots, x_n)$ based on W as

$$V = W + \tilde{W}, \tag{23}$$

where the functional $\tilde{W}$ is defined as

$$\begin{aligned} \tilde{W}(x_1, x_2, \ldots, x_n) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} |b_{ij}| \Bigg[\int_{t-\tau_{ij}}^{t} \int_{\theta}^{t} z_{ij}(\xi)\,\mathrm{d}\theta\,\mathrm{d}\xi \\ + \tau_{ij} \sum_{s=i-1}^{i+1} |b_{js}| \int_{t-\tau_{ij}}^{t} f^2(x_s(\xi))\,\mathrm{d}\xi \Bigg]. \end{aligned} \tag{24}$$

Calculating the upper right derivative $\mathrm{D}^+\tilde{W}$ of $\tilde{W}$ along the solution of (5), we have

$$\begin{aligned} \mathrm{D}^+\tilde{W} &= \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} |b_{ij}| \Bigg[\tau_{ij} z_{ij}(t) - \int_{t-\tau_{ij}}^{t} z_{ij}(\xi)\,\mathrm{d}\xi \\ &\quad + \tau_{ij} \sum_{s=i-1}^{i+1} |b_{js}| [f^2(x_s(t)) - f^2(x_s(t-\tau_{ij}))] \Bigg] \\ &\leqslant \frac{1}{2} \sum_{i=1}^{n} \sum_{j=i-1}^{i+1} |b_{ij}| \Bigg[\tau_{ij} k f^2(x_j(t))/p_j^2 - \int_{t-\tau_{ij}}^{t} z_{ij}(\xi)\,\mathrm{d}\xi \\ &\quad + \tau_{ij} \sum_{s=i-1}^{i+1} |b_{js}| f^2(x_s(t)) \Bigg]. \end{aligned} \tag{25}$$

Since the activation function of neuron is $f(x_i) = \tanh(x_i)$, we know that

$$f(x_i) = x_i(t) f'(\xi_i), \tag{26}$$

where ξ_i is the value between 0 and $x_i(t)$.

From (17) we know that every solution of (5) is bounded, and $|x_i(t)| \leqslant L_i$ holds. Then, we have

$$0 < f'(L_i) \leqslant f'(\xi_i) \leqslant 1, \qquad \text{for} \quad |\xi_i| \leqslant L_i.$$

Thus, by (26) and the fact $0 \leqslant f'(s) \leqslant 1$ for any s, we can easily obtain

$$|x_i(t)| = \frac{|f(x_i(t))|}{f'(\xi_i)} \geqslant |f(x_i(t))|.$$

In addition, by using the inequality $0 < f'(L_i) \leqslant f'(\xi_i) \leqslant 1$ for $|\xi_i| \leqslant L_i$, we can obtain

$$|x_i(t)| = \frac{|f(x_i(t))|}{f'(\xi_i)} \leqslant \frac{|f(x_i(t))|}{f'(L_i)} = \frac{|f(x_i(t))|}{p_i},$$

where $p_i \equiv f'(L_i)$. Therefore, we have

$$|f(x_i(t))| \leqslant |x_i(t)| \leqslant \frac{|f(x_i(t))|}{p_i}. \tag{27}$$

In deriving (25) we have used the above facts. Then,

$$\begin{aligned}
\mathrm{D}^+V &\leqslant \sum_{i=1}^{n}\left\{\Phi_i f^2(x_i(t)) + \sum_{j=i-1}^{i+1}|b_{ij}|\frac{\tau_{ij}}{2}q_{ij}f^2(x_i(t))\right.\\
&\quad\left. + \frac{1}{2}\sum_{j=i-1}^{i+1}|b_{ij}|\left[\frac{\tau_{ij}k}{p_j^2}f^2(x_j(t)) + \tau_{ij}\sum_{s=i-1}^{i+1}|b_{js}|f^2(x_s(t))\right]\right\}\\
&= \sum_{i=1}^{n}\left\{\Phi_i + \frac{1}{2}\sum_{j=i-1}^{i+1}|b_{ij}|\tau_{ij}q_{ij} + \frac{1}{2}\sum_{j=i-1}^{i+1}|b_{ji}|\frac{\tau_{ji}k}{p_i^2}\right.\\
&\quad\left. + \frac{1}{2}\sum_{s=i-1}^{i+1}\sum_{j=s-1}^{s+1}|b_{sj}||b_{ji}|\tau_{sj}\right\}f^2(x_i(t))\\
&= \sum_{i=1}^{n}\mu_i f^2(x_i(t)),
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
\mu_i &= \Phi_i + \frac{1}{2}\left[\sum_{j=i-1}^{i+1}(|b_{ij}|\tau_{ij}q_{ij}) + \sum_{j=i-1}^{i+1}\left(|b_{ji}|\,\tau_{ji}k/p_i^2\right) + \sum_{s=i-1}^{i+1}\sum_{j=s-1}^{s+1}(|b_{sj}||b_{ji}|\tau_{sj})\right]\\
&= -k + b_{ii} + \frac{1}{2}\sum_{j=i-1,j\neq i}^{i+1}(|b_{ij}| + |b_{ji}|) + \alpha_i.
\end{aligned}$$

Therefore, if $\max_{1\leqslant i\leqslant n}\mu_i < 0$, then we have $\mathrm{D}^+V \leqslant \sum_{i=1}^{n}\mu_i f^2(x_i(t)) < 0$. It is a consequence of (28) that $V(t) \leqslant V(0)$. Note that $x_i(t)$ is bounded on $[-\max_{1\leqslant i,j\leqslant n}\tau_{ij}, \infty)$, and thus $\dot{x}_i(t)$ is bounded on $[-\max_{1\leqslant i,j\leqslant n}\tau_{ij}, \infty)$. This, together with $x_i(t) \in L^2([0,\infty))$, implies (by a theorem of Gopalsamy [39]) that $\lim_{t\to\infty}x_i(t) = 0$. This completes the proof.

Corollary. *If* $\max_i\{-k + b_{ii} + \frac{1}{2}\sum_{j=i-1,j\neq i}^{i+1}(|b_{ij}| + |b_{ji}|\} < 0$, *then the equilibrium of network remains globally asymptotically stable when the time delays are small enough.*

In this section, we have investigated the global stability criteria depending upon all the parameters k, b_{ij}, τ_{ij} $(i, j = 1, 2, \ldots, n)$ and give the whole range of the parameter space in which the system is stable. Condition (18) gives a single relation of k, b_{ij}, τ_{ij} $(i, j = 1, 2, \ldots, n)$, which define a hypersurface in the parameter space. Generally, this hypersurface is complicated. In order to describe this hypersurface, we can fix some parameter and consider its intersection. Since there are cases that delay plays the key roles in discussion of whether or not a system is stable, we consider its intersection with k = constant, b_{ij} = constant and emphasize the effect of the time delay τ_{ij} on the global stability. In a similar way, we can also fix τ_{ij} = constant and consider the effect of other parameters on the stability by using condition (18).

To demonstrate the above statement, let us consider the neural network with delays

$$\begin{cases}\dot{x}(t) = -\frac{1}{2}x(t) - \frac{1}{2}f\left(x\left(t - \frac{1}{2}\tau\right)\right) + \frac{1}{4}f(y(t-\tau))\\ \dot{y}(t) = -\frac{1}{2}y(t) - \frac{1}{2}f\left(y\left(t - \frac{1}{2}\tau\right)\right) - \frac{2}{3}f(x(t-\tau)).\end{cases} \tag{29}$$

It can be verified that the neural network model does not satisfy the assumptions of theorem 1. The criteria given in theorem 1 fails to determine the global stability. However,

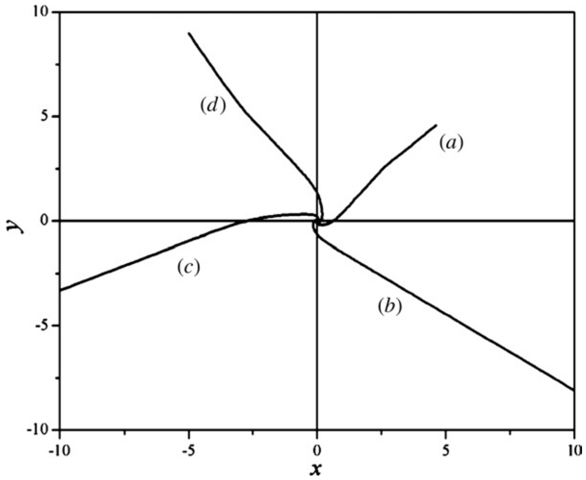

Figure 2. The graph of orbit of (29) with $\tau = 0.6$. The initial conditions are (a) (4.638, 4.58); (b) (20, −16); (c) (−30, −12); (d) (−5, 9) for $t \in [-\tau, 0]$.

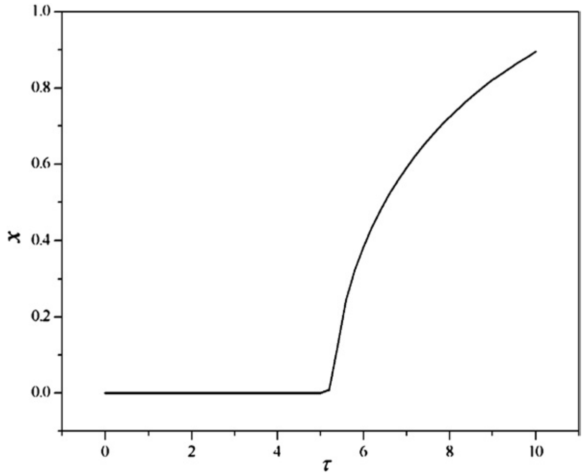

Figure 3. Bifurcation diagram of (29) on the Poincaré section $\sum = \{(\tau, x)|\ y = 0, \dot{y} > 0\}$.

employing (18) we can choose proper time delays to stabilize the system. Using Maple mathematical software, it is easy to compute that the parameters in (29) satisfy the conditions of theorem 2 when $\tau < 0.66$. Therefore, when $\tau = 0.6 < 0.66$, the solutions converge to the origin no matter whatever the initial dates are. The solution curves for different initial date are illustrated in the x–y plane as shown in figure 2.

Furthermore, we examine the dependence of stability on the delays. Assume that initial dates are $\phi_1(s) \equiv 4.6, \phi_2(s) \equiv -8.58, s \in [-\tau, 0]$. Figure 3 shows the bifurcation diagram of the system on the Poincaré section $\sum = \{(\tau, x)|\ y = 0, \dot{y} > 0\}$. It can be seen that when τ is large enough (>5.2), zero is unstable. Therefore, the global stability criteria presented by theorem 2 is conservative. It is an open problem how to estimate the precise upper bound of τ^* guaranteeing global stability.

3. Local existence of periodic solutions

The global stability criteria presented above may be conservative because the use of the Lyapunov's method depends sharply on the inequality estimation. In this section, the local stability analysis for system (5) is made so as to obtain some elaborating results. To simplify the analysis and computation, $\tau_{ij} = \tau$ is assumed to be true. In this case, (5) can be written as

$$\dot{x}_i = -kx_i(t) + \sum_{j=i-1}^{i+1} b_{ij} f(x_j(t-\tau)), \qquad i = 1, 2, \ldots, n. \tag{30}$$

The fact $f'(0) = 1$ results in the following characteristic matrix of the linearization of (30) at zeros:

$$\Delta(\tau, \lambda) = (\lambda + k)\mathbf{I} - \mathbf{B}\exp(-\lambda\tau). \tag{31}$$

The associated characteristic equation of (31) takes the form

$$\det\Delta(\tau, \lambda) = \det[(\lambda + k)\mathbf{I} - \mathbf{B}\exp(-\lambda)] = 0, \tag{32}$$

where $\mathbf{I}$ denotes the identity matrix.

To discuss the distribution of characteristic roots of (32), we put $\chi = \exp(2\pi\mathrm{i}/n)$ and

$$v_m = (c_0, c_1\chi^{1m}, c_2\chi^{2m}, c_3\chi^{3m}, \ldots, c_{n-1}\chi^{(n-1)k})^{\mathrm{T}}, \qquad m = 0, 1, 2, \ldots, n-1, \tag{33}$$

where i is the imaginary unit, $c_0, c_1, \ldots, c_{n-1}$ are constant which are determined as follows.

Using (33) and (31), we can obtain

$$\begin{aligned}(\Delta(\tau, \lambda)v_m)_j &= (\lambda + k)\chi^{(j-1)m} - \mathrm{e}^{-\lambda\tau}[b_{j,j-1}c_{j-2}\chi^{(j-2)m} + \beta c_{j-1}\chi^{(j-1)m} + b_{j,j+1}c_j\chi^{jm}] \\ &= \left\{\lambda + k - \exp(-\lambda\tau)\left[\frac{b_{j,j-1}c_{j-2}}{c_{j-1}}\chi^{-m} + \beta + \frac{b_{j,j+1}c_j}{c_{j-1}}\chi^m\right]\right\}c_{j-1}\chi^{(j-1)m}.\end{aligned} \tag{34}$$

It is easy to see that if

$$\frac{b_{21}c_0}{c_1} = \frac{b_{32}c_1}{c_2} = \cdots = \frac{b_{j,j-1}c_{j-2}}{c_{j-1}} = \frac{b_{j+1,j}c_{j-1}}{c_j} = \cdots \equiv \beta_1 \tag{35}$$

and

$$\frac{b_{12}c_1}{c_0} = \frac{b_{23}c_2}{c_1} = \cdots = \frac{b_{j-1,j}c_{j-1}}{c_{j-2}} = \frac{b_{j,j+1}c_j}{c_{j-1}} = \cdots \equiv \beta_2, \tag{36}$$

then (34) can be written as

$$(\Delta(\tau, \lambda)v_m)_j = \{\lambda + k - \exp(-\lambda\tau)[\beta_1\chi^{-m} + \beta + \beta_2\chi^m]\}(v_m)_j. \tag{37}$$

If connection strengths between the neurons satisfy

$$b_{j+1,j+2}b_{j+2,j+1} = b_{j+1,j}b_{j,j+1}, \qquad j = 1, 2, \ldots, n, \tag{38}$$

then we can from (35) and (36) obtain that

$$(c_j)^2 = \frac{b_{j+1,j+2}}{b_{j,j+1}}c_{j-1}c_{j+1} \tag{39}$$

and

$$c_1^n = \frac{c_0^n}{b_{12}^n}\alpha, \tag{40}$$

where $\alpha = \prod_{j=1}^n b_{j,j+1}$.

Therefore, we can compute

$$\beta_1 = b_{21}c_0\sqrt[n]{\frac{b_{12}^n}{c_0^n\alpha}}, \qquad \beta_2 = \frac{b_{12}}{c_0}\sqrt[n]{\frac{c_0^n\alpha}{b_{12}^n}}. \tag{41}$$

For simplicity, c_0 can be determined as follows.

(i) For n even and $\alpha > 0$, or n odd, we can choose $c_0 = 1$.
(ii) For n even and $\alpha < 0$, we can choose $c_0 = \sqrt[n]{-1}$.

Thus, we have the following results.

Lemma 1. *Assume (38) is satisfied. If* c_j $(j = 0, 1, 2, \ldots, n-1)$ *in (33) satisfy (39), then*

$$\det \Delta(\tau, \lambda) = \prod_{m=0}^{n-1} \left\{\lambda + k - \mathrm{e}^{-\lambda\tau}\left[\beta + (\beta_1+\beta_2)\cos\left(\frac{2m\pi}{n}\right) + \mathrm{i}(\beta_2-\beta_1)\sin\left(\frac{2m\pi}{n}\right)\right]\right\}, \tag{42}$$

where i *is the imaginary unit. To analyze the distribution of zeros of the characteristic equation (42), the following lemma plays an important role.*

Lemma 2. *Let* $z = R\mathrm{e}^{\mathrm{i}\theta}$, $0 \leqslant \theta < 2\pi$, *and consider*

$$q(\lambda) = \lambda + k - z\exp(-\lambda\tau). \tag{43}$$

(i) If $R \leqslant k$, *then* $q(\lambda)$ *has no purely imaginary zero of all* $\tau \geqslant 0$.
(ii) If $R > k$, *then there exists* $\tau_j := [\theta - \arccos(k/\mathrm{R}) + 2j\pi]/\sqrt{R^2-k^2} > 0$ *for any integer* j *such that* $q(\lambda)$ *has one and only one pair of purely imaginary zeros* $\pm\mathrm{i}\sqrt{R^2-k^2}$ *at* $\tau = \tau_j$, *and has no pair of purely imaginary zeros if* $0 < \tau \neq \tau_j$ *for such* j.
(iii) If $R > k$ *and* $\tau_j > 0$ *for some* j, *then there exist a sufficiently small* $\delta > 0$ *and a smooth curve* $\lambda : (\tau_j - \delta, \tau_j + \delta) \to C$ *such that* $q(\lambda(\tau)) = 0$ *for all* $\tau \in (\tau_j - \delta, \tau_j + \delta)$, $\lambda(\tau_j) = \mathrm{i}\sqrt{R^2-k^2}$ *and* $(\mathrm{d}/\mathrm{d}\tau)\,\mathrm{Re}(\lambda(\tau))|_{\tau=\tau_j} > 0$.

The proof of lemma 2 is similar to that of lemma 4.1 in [40], and thus is omitted. Applying lemma 2 to each factor of $\det \Delta(\tau, \lambda)$, *we get the following result.*

Lemma 3. *Assume (38) holds true.*

(i) If

$$R \equiv \left|\beta + (\beta_1+\beta_2)\cos\left(\frac{2m\pi}{n}\right) + \mathrm{i}(\beta_2-\beta_1)\sin\left(\frac{2m\pi}{n}\right)\right|^{1/2} \leqslant k, \tag{44}$$

for all $m = 0, 1, 2, \ldots, n-1$, *then all zeros of* $\det \Delta(\tau, \lambda)$ *have negative real parts.*

(ii) Assume there exists some $m \in \{0, 1, 2, \ldots, n-1\}$ *such that* $R > k$. *Define critical values*

$$\sigma_{m,j} = \frac{1}{\omega_m}[\theta - \arccos(k/\mathrm{R}) + 2j\pi], \qquad \text{for all} \quad j \in \mathbf{N}_0, \tag{45}$$

where $\omega_m = \sqrt{R^2-k^2}$ *and* $\theta = \mathrm{Arg}\left[\beta + (\beta_1+\beta_2)\cos\left(\frac{2m\pi}{n}\right) + \mathrm{i}(\beta_1+\beta_2)\sin\left(\frac{2m\pi}{n}\right)\right]$. *Then, at and only at* $\sigma_{m,j} (j \in \mathbf{N}_0)$, *(42) has purely imaginary eigenvalues* $\pm\mathrm{i}\omega_m$. *Moreover, for each fixed* $j \in \mathbf{N}_0$, *there exists* $\delta_{m,j} > 0$ *and* C^1*-mapping* $\lambda_{m,j} : (\sigma_{m,j} - \delta_{m,j}, \sigma_{m,j} + \delta_{m,j}) \to \mathbf{C}$ *such that* $\lambda_{m,j}(\sigma_{m,j}) = \mathrm{i}\omega_m$ *and* $\det[\Delta(\tau, \lambda_{m,j}(\tau))] = 0$ *for all* $\tau \in (\sigma_{m,j} - \delta_{m,j}, \sigma_{m,j} + \delta_{m,j})$. *Moreover,* $(\mathrm{d}/\mathrm{d}\tau)\,\mathrm{Re}\,\lambda_{m,j}(\sigma_{m,j}) > 0$.

Lemmas 2 and 3, together with the fact that the zero solution of (30) is uniformly asymptotically stable if and only if all zeros of $\det \Delta(\tau, \lambda)$ have negative real parts and that the zero solution of (30) is unstable if $\det \Delta(\tau, \lambda)$ has at least one zero with positive real part, lead to the following stability results.

Theorem 3. *Assume (38) is satisfied. For* $m \in \{0, 1, 2, \ldots, n-1\}$, *let* R *be defined by (44).*

(i) If $R \leqslant k$ *for all* $m = 0, 1, 2, \ldots, n-1$, *then the trivial equilibrium of (30) is locally stable for any* $\tau \geqslant 0$.

(ii) *Assume there exists* $m \in \{0, 1, 2, \ldots, n-1\}$ *such that* $R > k$ *and* $\beta + (\beta_1 + \beta_2)\cos\left(\frac{2m\pi}{n}\right) > k$, *then for all* $\tau \geqslant 0$, *the trivial equilibrium of system (30) is unstable.*

(iii) *Assume that* $R > k$ *and* $\beta + (\beta_1 + \beta_2)\cos\left(\frac{2m\pi}{n}\right) < k$ *for all* m. *Let* $\tau_0 = \min_{m,j}\{\sigma_{m,j}\}$. *Then the trivial equilibrium of (30) is locally asymptotically stable for* $\tau \in [0, \tau_0)$. *However, for* $\tau > \tau_0$ *the trivial equilibrium of system (30) is unstable.*

The time delay τ_0 from (45) is the critical value guaranteeing that the origin is stable when the original values of the system are near to the origin. However, theorem 2 and condition (18) give the global stability criteria for (5) when the original values are far away from the origin. These criteria, which are based on the Lyapunov function method, involve complicated inequality techniques and are usually conservative. Generally speaking, the maximum τ that satisfies (18) is obviously less than the time delay τ_0 obtained from theorem 3.

To compare the two delays, let us consider the system with $b_{i,i-1} = b_{i,i+1} = b = -0.1$, $\tau_{ij} = \tau$ and $k = 1$. It is easy to see that the system satisfies both conditions (18) and item (iii) in theorem 3. In this case, we can obtain from (18) that the origin of the system is globally stable when $0 < \tau < 0.033$. Moreover, from (45) we can also obtain the system with four neurons is locally stable when $0 \leqslant \tau_0 < 5.9$. The comparison shows that τ satisfying (18) is obviously less than that from (45).

4. Global existence of periodic solutions

From section 3, we can see that system (30) admits a Hopf bifurcation when τ crosses critical values $\sigma_{m,j}$, and periodic solutions can bifurcate from the trivial equilibrium. Usually, these periodic solutions only exist in a small neighborhood of the critical values. To extend the local Hopf bifurcation for large delay values, we investigate the global existence of these nontrivial periodic solutions by using a global Hopf bifurcation result [40].

In this section, we make the following assumption.

(**H**). The origin is the unique equilibrium of system (30).

Theorem 4. *Assume that (38) holds true. If there exists some* $m \in \{0, 1, 2, \ldots, n-1\}$ *such that* $R > k$, *then there exist critical time delays* $\sigma_{m,j}(j \in \mathbf{N}_0)$ *such that the Hopf bifurcation occur at these critical time delays. Moreover, if the parameters of (30) satisfy the conditions*

$$\max_{1\leqslant i\leqslant n}\left\{-k + \beta + \frac{1}{2}\sum_{j=i-1,j\neq i}^{i+1}(|b_{ij}| + |b_{ji}|)\right\} < 0, \tag{46}$$

then for given $m \in \{0, 1, 2, \ldots, n-1\}$ *(30) has at least* $j+1$ *periodic solutions for each* $\tau > \sigma_{m,j}$ $(j \geqslant 1)$, *where* R *and* $\sigma_{m,j}$ *are defined in lemma 3.*

The proof of theorem 4 can be found in the appendix.

To demonstrate the above statement, let us consider the neural network with delays modeled by

$$\begin{cases} \dot{x}(t) = -x(t) - f(x(t-\tau)) - 1.2 f(y(t-\tau)) \\ \dot{y}(t) = -y(t) - f(y(t-\tau)) + 1.5 f(x(t-\tau)). \end{cases} \tag{47}$$

It is easy to compute two series of critical values $\tau_0 = \sigma_{0,0} \approx 0.955$, $\sigma_{0,1} \approx 5.638$, $\sigma_{0,2} \approx 10.32$, and $\sigma_{1,0} \approx 2.342$, $\sigma_{1,1} \approx 7.025$, $\sigma_{1,2} \approx 11.708, \ldots$. By theorem 3, the trivial equilibrium is stable for $\tau \in [0, \tau_0)$ and a periodic solution bifurcates from the equilibrium as τ crosses τ_0 to the right, as shown in figure 4. Generally speaking, these periodic solutions

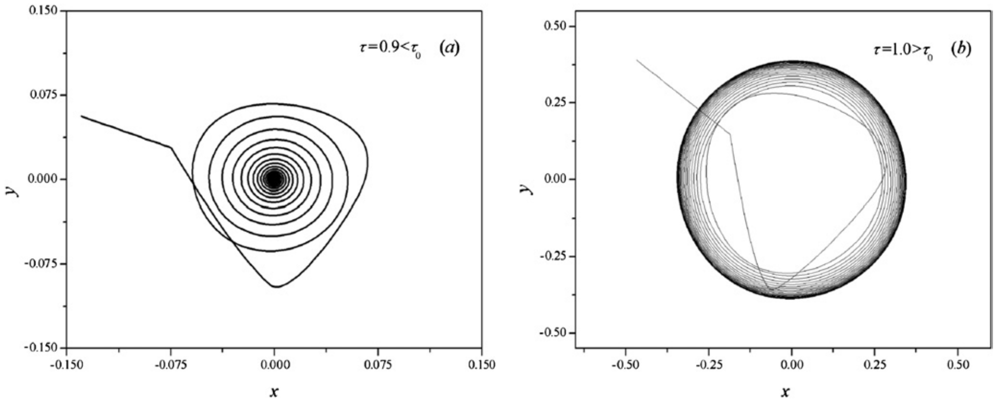

Figure 4. The solution of (30) near the critical time delay $\tau_0 = 0.955$.

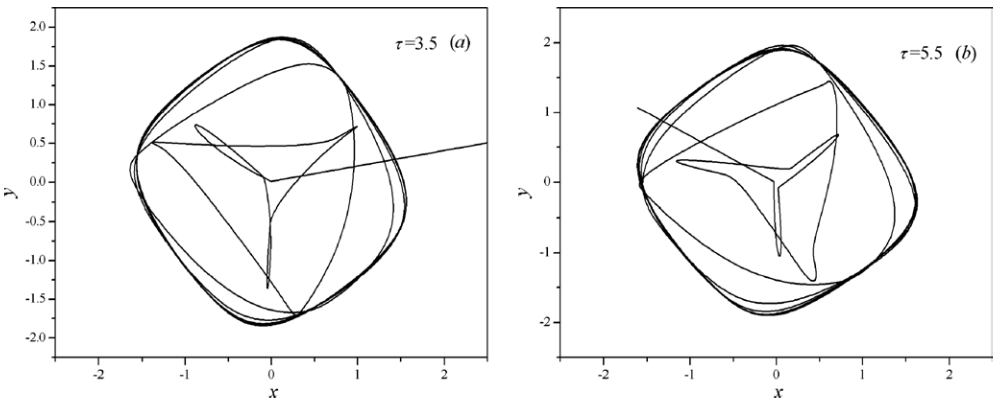

Figure 5. The periodic solutions still exist when τ is far away from τ_0.

only exist in a small neighborhood of the critical value. However, figure 5 shows that the periodic solutions still exist when the delay τ is far away from the critical values.

Figure 6 illustrates periodic solutions for (30) when τ is near to τ_0. The curves in figure 6(*a*) denote the solutions orbits in the *x*–*y* plane at critical delays $\sigma_{0,0}$, $\sigma_{0,1}$, $\sigma_{0,2}$, $\sigma_{0,3}$, $\sigma_{0,4}$ and $\sigma_{0,5}$, respectively; while figure 6(*b*) shows the periodic solution on the Poincaré section defined by $\sum = \{(\tau, x) | \, y = 0, \dot{y} > 0\}$, where each of the solid points denotes a periodic solution. It can be seen that when $\tau \in (\sigma_{0,1}, \sigma_{0,2})$ there exist two periodic solutions; while there are three periodic solutions for $\tau \in (\sigma_{0,2}, \sigma_{0,3})$ and four periodic solutions for $\tau \in (\sigma_{0,3}, \sigma_{0,4})$. These figures indicate that the system has at least $j + 1$ periodic solutions when $\tau > \sigma_{0,j} \geqslant \sigma_{0,1}$.

5. Routes to chaos

For the single-directed ring neural network [23] or the ring neural network with symmetrical weight matrix [24, 25, 27], it is very difficult to exhibit the much more complicated dynamics when the activation function is monotonic behavior because the structure of the network is

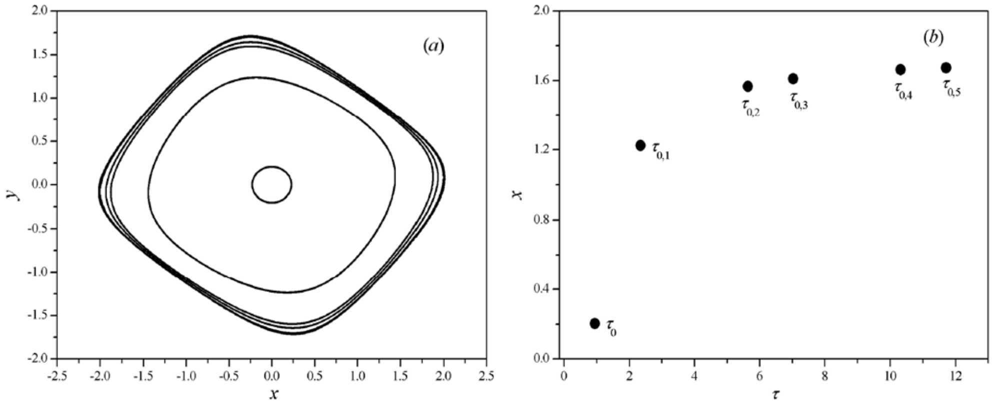

Figure 6. Global existence of periodic solutions for (30).

simple. However, system (5) describes a ring network with asymmetrical weighted matrix, and can exhibit complicated nonlinear dynamical behavior. In this section, the complicated dynamics will reveal numerically with the help of the package XPP [41].

Example 1. Consider the neural network with time delay

$$\begin{cases} \dot{x}(t) = -x(t) + 1.19f(x(t-\tau)) - 1.6f(y(t-\tau)) - 0.01f(z(t-\tau)) \\ \dot{y}(t) = -y(t) + 1.19f(y(t-\tau)) + 1.2f(x(t-\tau)) + 0.9f(z(t-\tau)) \\ \dot{z}(t) = -z(t) + 1.19f(z(t-\tau)) - 0.5f(x(t-\tau)) + 2.25f(y(t-\tau)). \end{cases} \tag{48}$$

In this example, τ is taken as a control parameter. In simplicity, the projection of solution curves onto the x–y plane is considered. Figure 7(a) shows coexisting periodic 1 solution when $\tau = 0.7$, in this case there exist two separate periodic orbits starting from different initial conditions. When $\tau = 0.85$ and $\tau = 0.87$, both periodic 2 and periodic 4 solutions appear as shown in figures 7(b) and (c), respectively. When τ is slightly increased to $\tau = 0.98$, two coexisting separate chaotic attractors with different initial conditions are observed in figure 7(d). Note that there are two separate single-scroll-like attractors in the figure, although it looks like a whole one. As further increase of time delay, two separate single-scroll-like attractors merge to a double-scroll-like one, such an attractor is displayed in figure 7(e) with $\tau = 1$. Obviously, it is the sequence of period-doubling bifurcations that leads to chaos.

To identify the routes to chaos, the Poincaré map is used. A Poincaré section is defined as a projection of solutions of system (5). The points in the Poincaré section depend on the behavior of the system. If the final motion of the system is periodic, there is only one point in the Poincaré section. For a period-n ($n = 2, 3, \ldots$) motion, n points will appear in the Poincaré section. For non-periodic motions such as a chaotic response, the number of points becomes infinite. An irregular pattern in the Poincaré section indicates the existence of a strange attractor.

Figure 8 illustrates the detailed bifurcation diagram as a function of the time delay τ by using the Poincaré section techniques which are defined as $\sum = \{(\tau, x)\colon (y = 0, \dot{y} > 0)\}$. It is a scenario of dynamics of system (5) with time delay increasing for the different original conditions. It shows that there exist two separated period-doubling bifurcation processes. The two period-doubling processes will have wrapping regions for the larger τ. In these wrapping regions, (48) has more complicated chaotic attractors illustrated in figures 9(a) and (b).

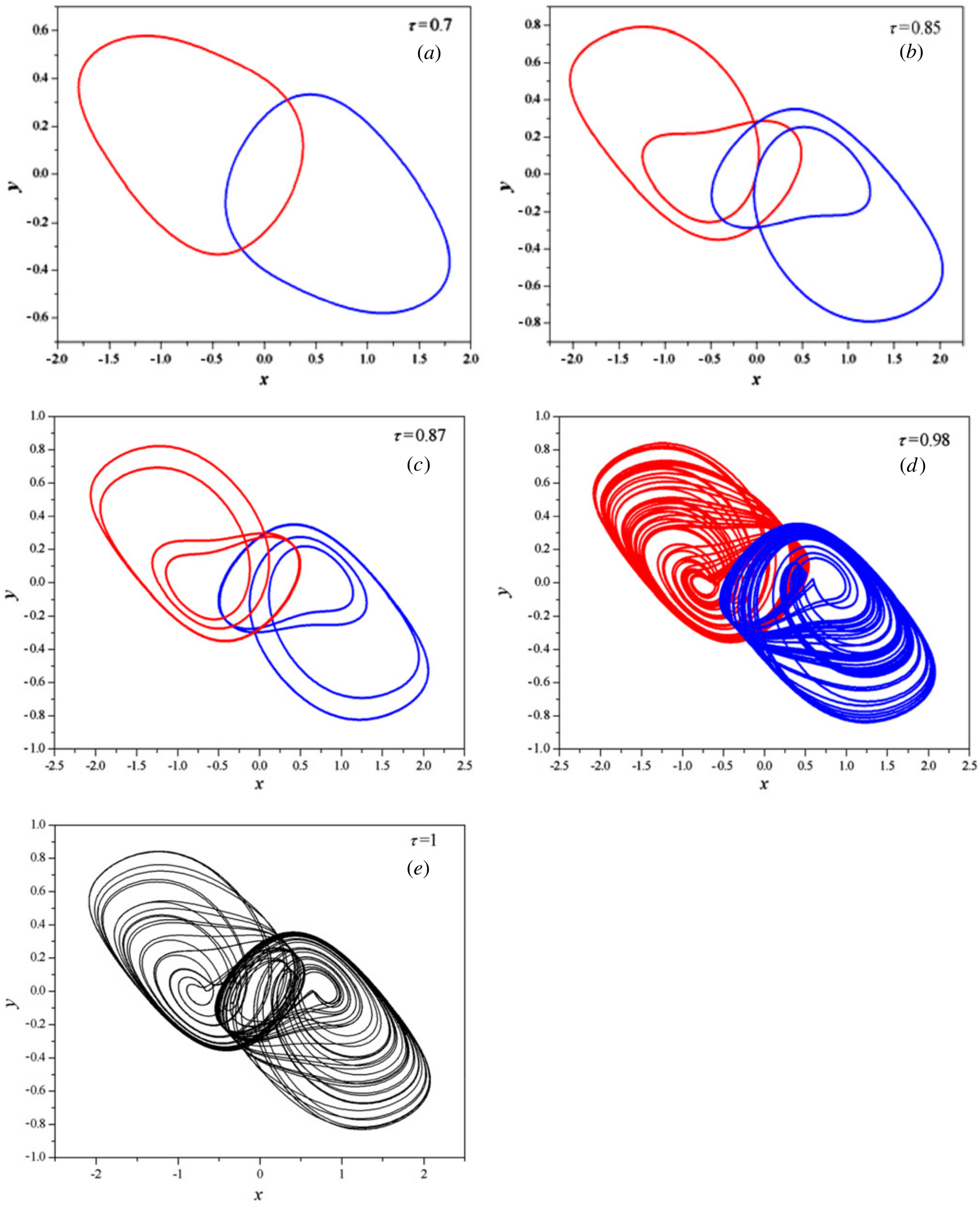

Figure 7. Periodic and chaotic solutions of (48) at sampled values of the time delay τ. Parts (a)–(c) show that the system has two coexisting period-1, period-2 and period-4 solutions. Part (d) shows two coexisting separated chaotic attractors for the different initial conditions. Obviously, it is the sequence of period-doubling bifurcations that leads to chaos.

Example 2. Consider the ring neural network with time delay

$$\begin{cases} \dot{x}(t) = -x(t) + 1.19 f(x(t-0.6)) - 1.6 f(y(t-0.6)) - 0.01 f(v(t-2)) \\ \dot{y}(t) = -y(t) + 1.19 f(y(t-0.6)) + 1.2 f(x(t-0.3)) + 0.9 f(z(t-1)) \\ \dot{z}(t) = -z(t) + 1.19 f(z(t-1)) + 0.5 f(y(t-0.6)) - 0.5 f(v(t-2)) \\ \dot{v}(t) = -v(t) + 1.19 f(v(t-2)) + A_{41} f(x(t-0.3)) + 0.425 f(z(t-1)), \end{cases} \tag{49}$$

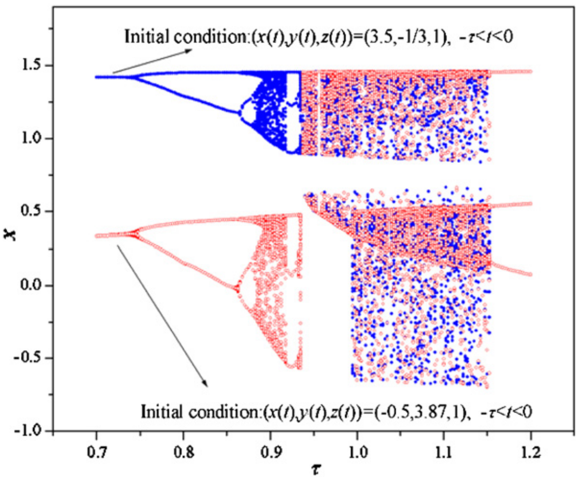

Figure 8. Bifurcation diagram of (48) on the Poincaré section for different initial conditions.

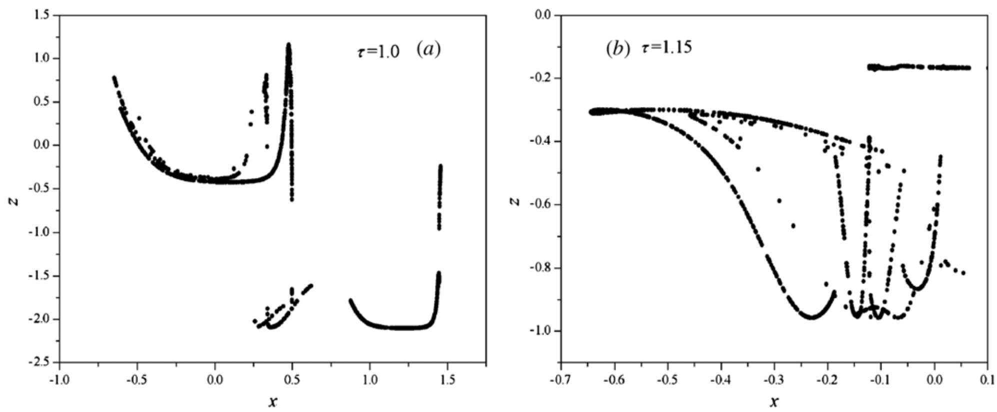

Figure 9. Strange chaotic attractors on the Poincaré section $\sum = \{(x, z)\colon (y(t) = 0, \dot{y}(t) > 0)\}$.

where A_{41} is taken as a control parameter and other parameters are fixed. The phase-plane and Poincaré section plots are used to locate the periodic and chaotic solutions.

When $A_{41} = -0.4$, the solution curve in the z–v phase-plane is first plotted as shown in figure 10(a), at which the motions fill the surface of a torus. And a discrete cluster of points in the Poincaré section $\sum = \{[z(t-1), z(t)]\colon [v(t) = 0, \dot{v}(t) > 0]\}$ closed up to form a loop as shown in figure 10(b). Such a motion is called quasi-periodic motion, as the ratio of the two frequencies becomes an irrational number. When the parameter A_{41} is slightly increased to $A_{41} = -0.1$, a stable period-3 solution is also observed and three points occur in the Poincaré section which is shown in figures 10(c) and (d). It is well known that the periodic-3 solutions indicate the chaos. When A_{41} is increased further, a strong attractor occurs and the closed curve in the Poincaré section breaks up into irregular patterns, indicating the formation of chaotic attractors, see figures 10(e) and (f). This phenomenon indicates that the chaos can directly occur through the bifurcations from the quasi-periodic solutions.

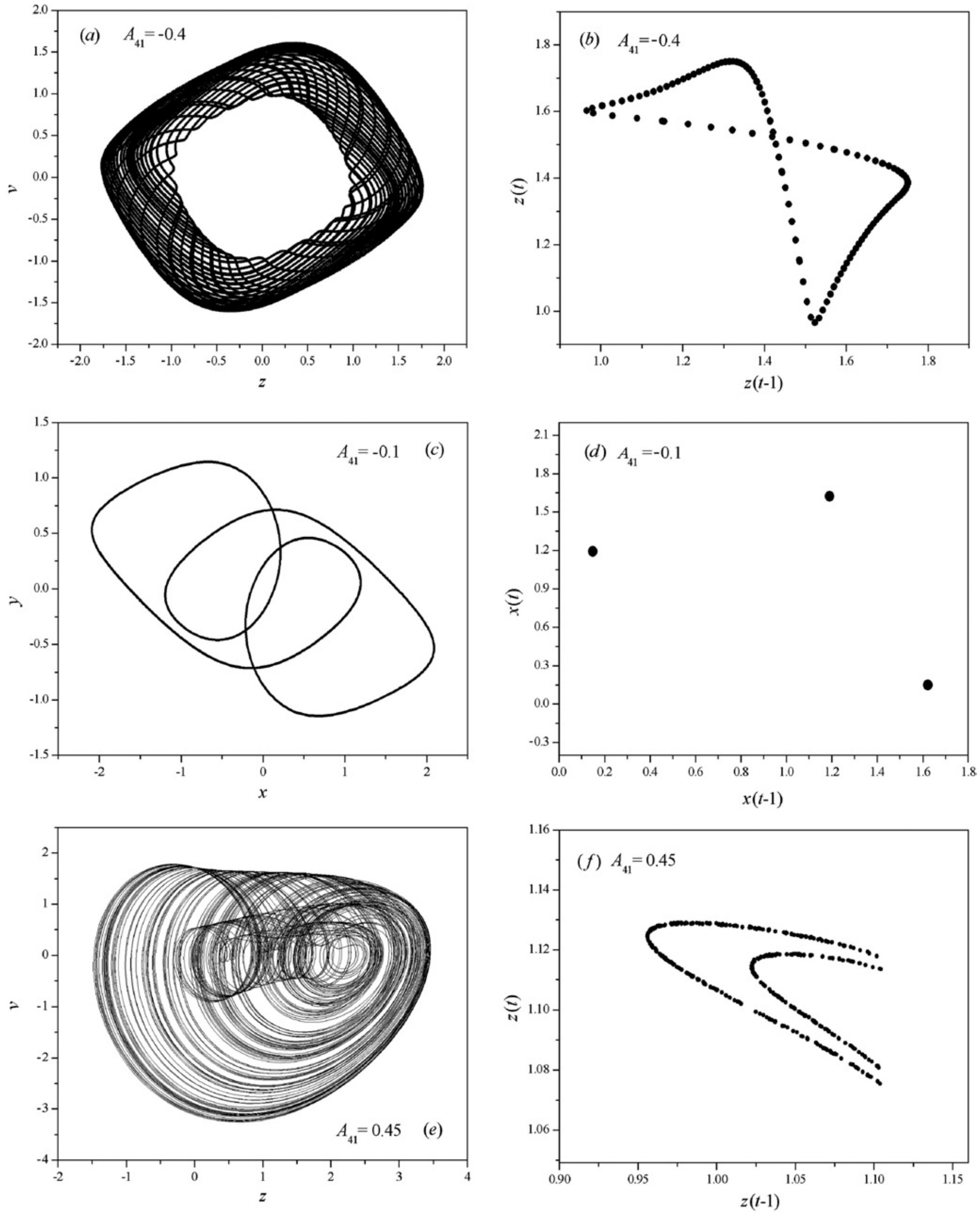

Figure 10. Phase-plane and Poincaré section plots exhibiting the process from quasi-periodic motions to chaos. (*a*) and (*b*): quasi-periodic attractor; (*c*) and (*d*): periodic-3 attractor; (*e*) and (*f*): chaotic attractor.

The above numerical results show that there exist two routes to chaos in network (5) for different network structures and parameters. One is from the period-doubling sequences to chaos, and the other is from the quasi-periodic solutions to chaos. For the latter case, the occurrence of chaos can be roughly considered as the torus breaking or quasi-periodic solutions going into the phase-locked wrapping regions with different frequency.

6. Conclusions

In this paper, we have studied the dynamics of a ring neural network with delays in detail. First, the delay-independent and delay-dependent criteria for global asymptotic stability are investigated based on the approach of the Lyapunov function. Our work generalizes that reported in [36, 37], and shows that when the delay-independent criteria for the global stability are not satisfied, we can choose proper time delays to globally stabilize the system. It indicates that the delay-dependent global stability criteria are less conservative and restrictive than the delay-independent criteria. In the mean time, we also note that if the ring neural network starts with a stable equilibrium, but then becomes unstable due to delays, it will likely be destabilized by means of a Hopf bifurcation leading to periodic solutions. Generally speaking, these obtained bifurcating periodic solutions only exist when the time delay is in a small neighborhood of the critical value. In this paper, we extend the scope of local periodic solutions, and obtain the existence of periodic solutions for time delay far away from the local Hopf bifurcation values. It shows that the local Hopf bifurcation implies the global Hopf bifurcation as the time delay is large. In addition, complicated dynamical behavior of (5) has been investigated with the help of numerical simulation. For the different networks, it may have two routes to chaos, such as the sequence of period-doubling bifurcations and the bifurcation from quasi-periodic solutions. The results show that the ring neural network exhibits the complicated dynamics from order to chaos or vice versa. This shall be the motivation of some further studies of the dynamics of the ring neural networks.

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Appendix

To prove theorem 4, we fist give some definitions and lemmas.

Let $\mathbf{X} = C([-\tau, 0], \Re^n)$ be the Banach space of bounded continuous mapping from $[-\tau, 0]$ into $\Re^n$ equipped with the supremum norm. For $z \in \mathbf{X}$, $t \in \Re$, define $z_t(s) = z(t+s)$ for $s \in \Re$. For simplification of notations, we rewrite (30) as the following functional differential equation:

$$\dot{z}(t) = F(z_t, \tau, T), \tag{A.1}$$

where T is the periodic of periodic solution. We introduce some notations:

$N = \{(\hat{z}, \tau, T)\colon \hat{z}$ is a fixed point$\}$, $\Sigma = C_l\{(z, \tau, T)\colon z$ is a T-periodic solution$\} \subset \mathbf{X} \times \Re_+ \times \Re_+$,

$$\Delta(\hat{z}, \tau, T)\,(\lambda(\tau)) = \prod_{m=0}^{n-1} \left\{\lambda + k - \exp(-\lambda\tau)\left[\beta + (\beta_1 + \beta_2)\cos\left(\frac{2m\pi}{n}\right) + \mathrm{i}(\beta_2 - \beta_1)\sin\left(\frac{2m\pi}{n}\right)\right]\right\}$$

and $\lambda(\hat{z}, \tau, T)$ denotes the connected component of $(\hat{z}, \tau, T)$ in Σ.

Consider (A.1) parameterized by two real numbers $(\tau, T) \in \Re_+ \times \Re_+$, where $\Re_+ = (0, \infty)$ and $F\colon X \times \Re_+ \times \Re_+ \to \Re^n$ is completely continuous. Identifying the subspace of **X** consisting of all constant mappings with $\Re^n$, we obtain a mapping $\hat{F} = F|_{\Re^n \times \Re_+ \times \Re_+}\colon \Re^n \times \Re_+ \times \Re_+ \to \Re^n$. Assume

(B1) $\hat{F}$ is twice continuously differentiable.
(B2) At each stationary solution $(\hat{z}_0, \tau_0, T_0)$, the derivative of $\hat{F}(z, \tau, T)$ with respect to the first variable z, evaluated at $(\hat{z}_0, \tau_0, T_0)$, is an isomorphism of $\Re^n$.
(B3) $F(\varphi, \tau, T)$ is differentiable with respect to φ, and the $n \times n$ complex matrix function $\Delta_{(\hat{z}(\tau,T),\tau,T)}(\lambda)$ is continuous in $(\tau, T, \lambda) \in B_{\varepsilon_0}(\tau_0, T_0) \times C$.
(B4) There exist $\varepsilon > 0, \delta > 0$ such that on $\tau \in [\tau_0 - \delta, \tau_0 + \delta] \times \partial\Omega_{\varepsilon,T_0}$, $\Delta_{(\hat{z}(\tau,T),\tau,T)}(u + \mathrm{i}s2\pi/T) = 0$ if and only if $\tau = \tau_0$, $u = 0$ and $T = T_0$, where $\Omega_{\varepsilon,T_0} = \{(u, T)\colon 0 < u < \varepsilon, |T - T_0| < \varepsilon\}$.

Lemma 4 [40]. *Assume that $(\hat{z}_0, \tau_0, T_0)$ is an isolated center satisfying the hypotheses (B1–B4). Denote by $\lambda_{(\hat{z}_0,\tau_0,T_0)}$ the connected component of $(\hat{z}_0, \tau_0, T_0)$ in Σ. Then*

(i) $\lambda_{(\hat{z}_0,\tau_0,T_0)}$ is unbounded, or
(ii) $\lambda_{(\hat{z}_0,\tau_0,T_0)}$ is bounded, $\lambda_{(\hat{z}_0,\tau_0,T_0)} \cap N$ is finite and

$$\sum_{(\hat{z},\tau,T)\in\lambda_{(\hat{z}_0,\tau_0,T_0)}\cap N} \Gamma_m(\hat{z}, \tau, T) = 0,$$

for all $s = 1, 2, \ldots,$ where $\Gamma_s(\hat{z}, \tau, T)$ is the sth crossing number of $(\hat{z}, \tau, T)$ if $s \in J(\hat{z}, \tau, T)$, or is zero if otherwise, where $J(\hat{z}, \tau, T)$ denotes the set of all positive integers s such that is $2\pi/T$ is a characteristic value of $(\hat{z}, \tau, T)$.

Lemma 5. *All the periodic nontrivial solutions of (30) are uniformly bounded if the connection strengths between neurons are bounded.*

Proof. Let $(x_1(t), x_2(t), \ldots, x_n(t))$ be a non-constant periodic solution of (30).

If $M_i = \max\{x_i(t)\} = x_i\left(t_i^M\right)$ and $N_i = \min\{x_i(t)\} = x_i\left(t_i^N\right)$, respectively, be the maximum and minimum values of the periodic function $x_i(t)$, then we have

$$kM_i = \sum_{j=i-1}^{i+1} b_{ij} f\left(x_j\left(t_i^M - \tau\right)\right), \qquad kN_i = \sum_{j=i-1}^{i+1} b_{ij} f\left(x_j\left(t_i^N - \tau\right)\right), \qquad i = 1, 2, \ldots, n. \tag{A.2}$$

Thus, (A.2) gives $k|M_i| \leqslant |\beta| + |b_{i,i+1}| + |b_{i,i-1}|$, $k|N_i| \leqslant |\beta| + |b_{i,i+1}| + |b_{i,i-1}|$. This completes the proof.

Lemma 6. *If parameters satisfy the inequality*

$$\max_{1\leqslant i\leqslant n} \left\{ -k + \beta + \frac{1}{2} \sum_{j=i-1, j\neq i}^{i+1} (|b_{ij}| + |b_{ji}|) \right\} < 0, \tag{A.3}$$

then (30) has no periodic solution of period τ.

Proof. If $(x_1(t), x_2(t), \ldots, x_n(t))$ is a non-constant periodic solution of periodic τ, then it is a periodic solution to the following ordinary equation:

$$\dot{x}_i = -kx_i(t) + \sum_{j=i-1}^{i+1} b_{ij} f(x_j(t)), \qquad i = 1, 2, \ldots, n. \tag{A.4}$$

For this system of ordinary differential equations, we consider the following Lyapunov function:

$$V(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \int_0^{x_i(t)} f(\xi)\,\mathrm{d}\xi. \tag{A.5}$$

The derivative V along the solution of (A.4) satisfies

$$\begin{aligned}\dot{V} &= \sum_{i=1}^{n} f(x_i(t))\dot{x}_i(t) = \sum_{i=1}^{n} f(x_i(t))\left\{-kx_i(t) + b_{ii} f(x_i(t)) + \sum_{j=i-1, j\neq i}^{i+1} b_{ij} f(x_j(t))\right\}\\ &\leqslant \sum_{i=1}^{n}\left\{-kf^2(x_i(t)) + b_{ii} f^2(x_i(t)) + \frac{1}{2}\sum_{j=i-1, j\neq i}^{i+1}(|b_{ij}| + |b_{ji}|) f^2(x_i(t))\right\}\\ &= \sum_{i=1}^{n}\left\{-k + \beta + \frac{1}{2}\sum_{j=i-1, j\neq i}^{i+1}(|b_{ij}| + |b_{ji}|)\right\} f^2(x_i(t)).\end{aligned} \tag{A.6}$$

By the well-known invariance principle, when inequality (A.3) is satisfied, each solution of (A.4) is convergent to equilibrium. Consequently, (A.4) (and thus (30)) has no non-constant periodic solutions.

The proof of theorem 4.

First, note that

$$F(z_t, \tau, T) := \left[-kx_1(t) + \sum_{j=0}^{2} b_{1j} f(x_j(t-\tau)), \ldots, -kx_n(t) + \sum_{j=n-1}^{n+1} b_{nj} f(x_j(t-\tau))\right]^{\mathrm{T}}$$

satisfies the hypotheses (B1–B4) with $(\hat{z}, \tau, T) = (\mathbf{0}, \bar{\tau}_j, 2\pi/\omega_0)$ and $\Delta_{(\mathbf{0},\bar{\tau}_j,2\pi/\omega_m)}(\lambda) = 0$.

It can be verified that $(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)$ is an isolated center. Moreover, putting

$$H^{\pm}(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)(\eta, T) = \Delta(\mathbf{0}, \sigma_{m,j} \pm \varepsilon, T)(\eta + \mathrm{i}2\pi/T).$$

We have the crossing number

$$\begin{aligned}&\Gamma(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)\\ &\quad= \deg_B(H^-(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)(\eta, T), \Omega_\varepsilon) - \deg_B(H^+(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)(\eta, T), \Omega_\varepsilon)\\ &\quad= -1.\end{aligned}$$

By assumption (H), origin is the only fixed point of (30), we have

$$\sum_{(\hat{z},\tau,T)\in\lambda(\mathbf{0},\sigma_{m,j},2\pi/\omega_m)} \Gamma(\hat{z}, \tau, T) < 0,$$

where $(\hat{z}, \tau, T)$ takes the form of $(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)$, $j = 0, 1, 2, \ldots$. It follows from lemma 4 that the connected component $\lambda(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)$ through $(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)$ in Σ is non-empty. And hence, $\lambda(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)$ is unbounded. From the definition of $\tau_{m,j}$ in lemma 3, we have $2\pi/\omega_m < \sigma_{m,j}$ for $j \geqslant 1$. And the periodic $T = 2\pi/\omega_m$ is bounded when $j \geqslant 1$. Clearly, lemma 6 shows that (30) with $\tau = 0$ has no nontrivial periodic solution. Hence, the projection of $\lambda(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)$ onto τ-space must be an interval $[\tilde{\tau}, +\infty)$ with $0 < \tilde{\tau} \leqslant \sigma_{m,j} (j \geqslant 1)$. This shows that for each $\tau > \sigma_{m,j} (j \geqslant 1)$, (30) has a non-constant periodic solution on $\lambda(\mathbf{0}, \sigma_{m,j}, 2\pi/\omega_m)$. Therefore, if $\tau > \sigma_{m,j} (j \geqslant 1)$, (30) has at least $j + 1$ periodic solutions. This completes the proof of theorem 4.

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